PHYSICS INDUCTION
An institute of Science & Mathematics

Vinita Chawla
M.Sc. Physics (HRC, Delhi University)
M.Sc. Computer Science, B.Ed.,
CTET, HTET, PSTET, CSB Qualified
Contact: 91-8168258349
physicsinduction@gmail.com

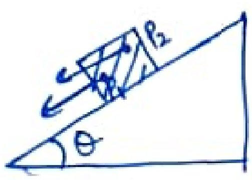
NOTES : CLASS XI: SYSTEM OF PARTICLES AND ROTATIONAL MOTION

RIGID BODY:- A rigid body is a body with definite shape & definite size. It consists of a collection of n number of particles interacting with one another.

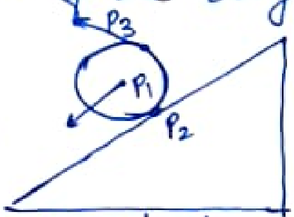
Internal forces:- Forces exerted by particles on one another are mutual i.e., equal & opposite ∴ cancel out in pairs. $(\vec{F}_{12} = -\vec{F}_{21})$.

External forces:- forces exerted by external agency. Overall motion of the system is affected by external forces only.

WHAT KIND OF MOTION CAN A RIGID BODY HAVE?

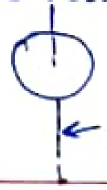


An Object sliding down an inclined Plane, is executing pure translational motion. At any instant of time, all particles of the body have the same velocity.



An object rolling down an inclined plane, executes rotational + translational motion.

All its particles aren't moving with the same velocity at any instant. Every particle of the body moves in a circle, about the axis of rotation.



CENTRE OF MASS:-

System $\xrightarrow{\text{replace}}$ Single point object.

A point at which the entire mass of the system/body is supposed to be concentrated.

If all external forces act on system were applied at C.M., its state (rest/motion) remain unaffected.

CENTRE OF MASS OF TWO PARTICLE SYSTEM:-

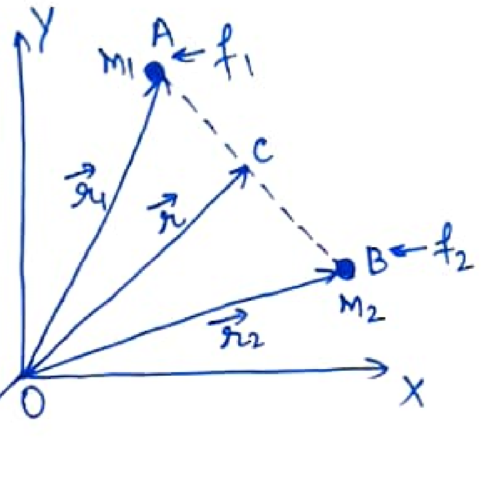
C: Position of C.M., $\vec{OC} = \vec{r}$

f_1 & f_2: External forces

$\vec{v}_1 = \dfrac{d\vec{r}_1}{dt}$, $\vec{v}_2 = \dfrac{d\vec{r}_2}{dt}$

$\vec{F}_{12}$ & $\vec{F}_{21}$: Internal forces.

$\vec{P}_1 = m_1\vec{v}_1$, $\vec{P}_2 = m_2\vec{v}_2$



for particle at A:-

$$\frac{d\vec{P}_1}{dt} = \vec{f}_1 + \vec{F}_{12} \text{ (Newton's 2nd law)}$$

for particle at B:-

$$\frac{d\vec{P}_2}{dt} = \vec{f}_2 + \vec{F}_{21}$$

$$\therefore \frac{d}{dt}(\vec{P}_1 + \vec{P}_2) = \vec{f}_1 + \cancel{\vec{F}_{12}} + \vec{f}_2 + \cancel{\vec{F}_{21}}$$

$$\Rightarrow \frac{d}{dt}(m_1\vec{v}_1 + m_2\vec{v}_2) = \vec{f} \text{ (say)}$$

$$\Rightarrow \frac{d}{dt}\left(m_1\frac{d\vec{r}_1}{dt} + m_2\frac{d\vec{r}_2}{dt}\right) = \vec{f}$$

$$\Rightarrow \frac{d}{dt}\left[\frac{d}{dt}(m_1\vec{r}_1 + m_2\vec{r}_2)\right] = \vec{f}$$

$$\Rightarrow \frac{d^2}{dt^2}(m_1\vec{r}_1 + m_2\vec{r}_2) = \vec{f}$$

$$\Rightarrow (m_1+m_2)\frac{d^2}{dt^2}\underbrace{\frac{(m_1\vec{r}_1 + m_2\vec{r}_2)}{(m_1+m_2)}}_{\vec{r}} = \vec{f} \quad \left\{\begin{matrix}\text{dividing \& multiplying by} \\ (m_1+m_2) \text{ on left side}\end{matrix}\right\}$$

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$$\Rightarrow (m_1+m_2)\frac{d^2\vec{r}}{dt^2} = \vec{f} \quad , \text{where } \vec{r} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{(m_1+m_2)}$$

Total mass of the system is supposed to be concentrated at a point, whose position vector is $\vec{r}$.

$$\frac{d^2\vec{r}}{dt^2} = \text{Acceleration}$$

$\vec{f}$: Motion of the total mass is described by external forces only, as internal forces cancel out in pairs.

$\vec{r}$: Weighted Avg. of the position of two particles.

Case-1: If C.M. is at origin, $\vec{r}=0 \Rightarrow \underbrace{\vec{r}_2}_{+ve} = -\frac{m_1}{m_2}\underbrace{\vec{r}_1}_{-ve}$

$\Rightarrow m_1$ is on the left of origin & m_2 on the right of origin. C.M. on the line joining these particles.

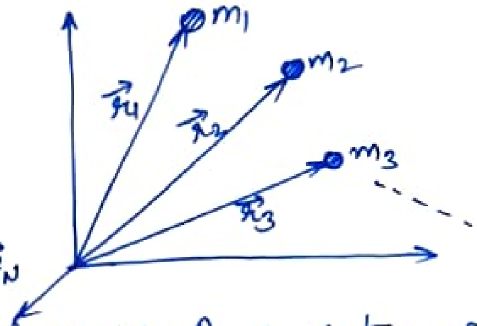
Case-2: If $m_1 = m_2 = m$ (say)

$\Rightarrow \vec{r} = \frac{\vec{r}_1+\vec{r}_2}{2} \Rightarrow$ Average of position vectors of 2 particles.

Case-3: If $m_1 > m_2 \Rightarrow \vec{r}_1 < \vec{r}_2$ i.e., C.M. lies closer to m_1.

$\Rightarrow$ C.M. divides internally the line joining the two particles in the inverse ratio of masses.

CENTRE OF MASS OF N-PARTICLES :-



Masses : $m_1, m_2, m_3, ------, m_N$

pos^n vectors : $\vec{r}_1, \vec{r}_2, \vec{r}_3, ------, \vec{r}_N$

External forces : $\vec{f}_1, \vec{f}_2, \vec{f}_3, ------, \vec{f}_N$

Instantaneous velocities : $\vec{v}_1, \vec{v}_2, \vec{v}_3, -----, \vec{v}_N$

Let $\vec{r}$ be the position vector of centre of mass of a system of N-particles.

Total Internal Force, $\vec{F}_i = \sum_{\substack{j=1 \\ j\neq i}}^{n} \vec{F}_{ij}$: Force on the i^{th} particle due to j^{th} particle

for i^{th} particle : Eq^n of motion :

$$\frac{d}{dt}(m_i\vec{v}_i) = \vec{F}_i + \vec{f}_i$$

for n-particles : $\sum_{i=1}^{n}\frac{d}{dt}(m_i\vec{v}_i) = \sum_{i=1}^{n}\vec{F}_i + \sum_{i=1}^{n}\vec{f}_i$ $\left\{\begin{array}{l}\text{But,} \\ \sum_i F_i = 0 \\ \sum_i f_i = f \text{ (say)}\end{array}\right.$

$$\Rightarrow \sum_{i=1}^{n}\frac{d}{dt}\left(m_i\frac{d\vec{r}_i}{dt}\right) = f$$

$$\Rightarrow \sum_{i=1}^{n}\frac{d^2}{dt^2}(m_i\vec{r}_i) = f$$

$$\Rightarrow \frac{d^2}{dt^2}\sum_{i=1}^{n} m_i\vec{r}_i = f$$

$$\Rightarrow M\frac{d^2}{dt^2}\underbrace{\sum_{i=1}^{n}\frac{m_i\vec{r}_i}{M}}_{\vec{r}} = f$$

where,

$$\underbrace{\vec{r} = \sum_{i=1}^{n}\frac{m_i r_i}{M} = \frac{m_1r_1+m_2r_2+----+m_Nr_N}{m_1+m_2+-----+m_N}}$$

COORDINATES OF C.M. :-

$$\vec{r} = \frac{\sum_{i=1}^{n} m_i \vec{r}_i}{M} \quad \text{(for a system of n-part)} \quad -①$$

position vector of ith particle : $\vec{r}_i = x_i\hat{i} + y_i\hat{j} + z_i\hat{k}$, $i = 1, 2, ---, n$

position vector of C.M: $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, (x, y, z - coord. of c.m.)

∴ ① b'comes,

$$(x\hat{i} + y\hat{j} + z\hat{k}) = \frac{1}{M}\sum_{i=1}^{n} m_i[x_i\hat{i} + y_i\hat{j} + z_i\hat{k}]$$

equating separately, we get.

$$x = \frac{1}{M}\sum_i m_i x_i = \frac{1}{M}(m_1x_1 + m_2x_2 + ------- + m_nx_n)$$

$$|||^{ly},\ y = \frac{1}{M}\sum_i m_i y_i = \frac{1}{M}(m_1y_1 + m_2y_2 + ------- + m_ny_n)$$

$$z = \frac{1}{M}\sum_i m_i z_i = \frac{1}{M}(m_1z_1 + m_2z_2 + ------- + m_nz_n)$$

Summation can be replaced by integration in case of irregular shapes/ non-uniform bodies.

Consider small element of mass, dm & coordinates (x, y, z).

Coordinates of CM :-

$$\boxed{X = \frac{1}{M}\int x\,dm \ , \quad Y = \frac{1}{M}\int y\,dm \ , \quad Z = \frac{1}{M}\int z\,dm}$$

MOTION OF CENTRE OF MASS :-

Velocity of CM :-

As, $\vec{r} = \dfrac{\sum_{i=1}^{n} m_i\vec{r}_i}{M} \Rightarrow M\vec{r} = \sum_{i=1}^{n} m_i\vec{r}_i$

Differentiating w.r.t time, we get

$$M\frac{d\vec{r}}{dt} = m_1\frac{d\vec{r}_1}{dt} + m_2\frac{d\vec{r}_2}{dt} + ------- + m_n\frac{d\vec{r}_n}{dt}$$

$$\Rightarrow M\vec{v} = m_1\vec{v}_1 + m_2\vec{v}_2 + ------- + m_n\vec{v}_n$$

$$\Rightarrow \boxed{\vec{v} = \frac{m_1\vec{v}_1 + m_2\vec{v}_2 + ------ + m_n\vec{v}_n}{M} = \frac{\sum_{i=1}^{n} m_i\vec{v}_i}{M}}$$

Acceleration of CM :-

As, $M\vec{v} = m_1\vec{v}_1 + m_2\vec{v}_2 + \cdots + m_n\vec{v}_n$

differentiating w.r.t time, we get

$$M\frac{d\vec{v}}{dt} = m_1\frac{d\vec{v}_1}{dt} + m_2\frac{d\vec{v}_2}{dt} + \cdots + m_n\frac{d\vec{v}_n}{dt}$$

$$\Rightarrow M\vec{a} = m_1\vec{a}_1 + m_2\vec{a}_2 + \cdots + m_n\vec{a}_n$$

$$\Rightarrow \boxed{\vec{a} = \frac{m_1\vec{a}_1 + m_2\vec{a}_2 + \cdots + m_n\vec{a}_n}{M} = \frac{\sum_{i=1}^{n} m_i\vec{a}_i}{M}}$$

LINEAR MOMENTUM OF A SYSTEM OF PARTICLES :-

The total Momentum of a system of particles is equal to the product of the total mass of the system & the velocity of its centre of mass.

As, $M\vec{v} = m_1\vec{v}_1 + m_2\vec{v}_2 + \cdots + m_n\vec{v}_n$

$$\Rightarrow \vec{P} = \vec{P}_1 + \vec{P}_2 + \cdots + \vec{P}_n$$

$$\vec{P} = M\vec{v}$$

$$\Rightarrow \frac{d\vec{P}}{dt} = M\frac{d\vec{v}}{dt} = M\vec{a} \quad , \quad \frac{d\vec{P}}{dt} = \vec{F}_{ext}$$

If $\vec{F}_{ext} = 0 \Rightarrow \frac{d\vec{P}}{dt} = 0 \Rightarrow \underline{\vec{P} = \text{constant}}$

$\Rightarrow M\vec{v} = \text{const} \Rightarrow \underline{\vec{v} = \text{const}}$

of C.M.

Consider a system of n, number of particles, under the action of total force, $\vec{f}$.

$$\vec{f} = \sum_{i=1}^{n} \frac{d(m_i\vec{v}_i)}{dt}$$

$\Sigma F_{int} = 0$ ∵ cancel out in pairs

If no external force acts, i.e., $f = 0$

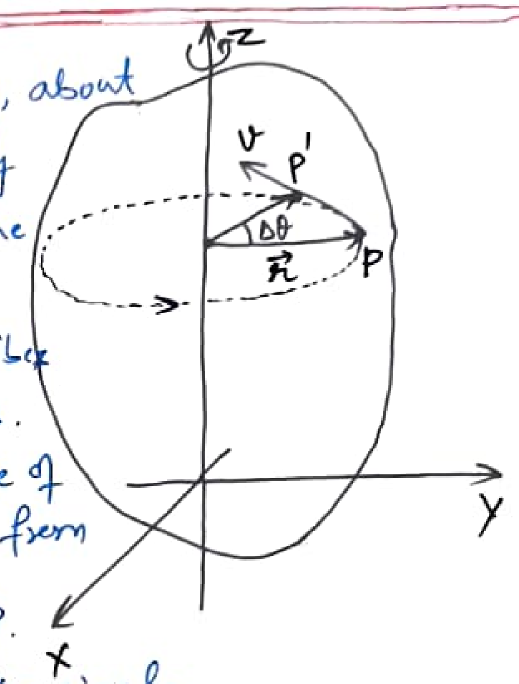
$$\Rightarrow \sum_{i=1}^{n} \frac{d}{dt}(m_i\vec{v}_i) = 0$$

$$\Rightarrow \sum_{i=1}^{n} (m_i\vec{v}_i) = \text{constant}$$

$$\Rightarrow \sum_{i=1}^{n} P_i = \text{Constant}$$

When the total external force, acting on a system of particles is zero, the total linear momentum of the system is constant.

ANGULAR VELOCITY AND ITS RELATION WITH LINEAR VELOCITY



In Rotational motion of a rigid body, about a fixed axis, every particle of the body moves in a circle, which lies in a plane perpendicular to the axis & has its centre on the axis. The particle describes a circle with a centre, C on the axis.

$\vec{r}$ = Radius of the circle → ⊥r distance of the pt., P from the axis.

$\vec{v}$: Linear Velocity of the particle at P.
It's along the tangent at P to the circle.

P' : Position of the particle after time, Δt

$\angle PCP' = \Delta\theta$: Angular displacement of the particle in time, Δt.

∴ Average angular velocity, ω over the interval, Δt :

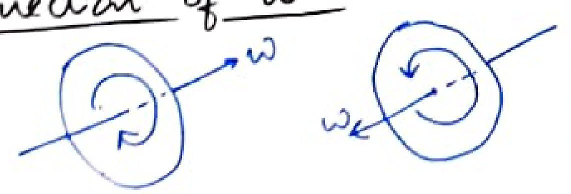
$$\boxed{\omega = \frac{\Delta\theta}{\Delta t}}$$

Instantaneous Angular velocity, $\boxed{\omega_{inst} = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}}$

For a particle at a perpendicular distance, r_i from the fixed axis, the linear velocity at a given instant, v_i is given by : $v_i = \omega r_i$ $\quad i = 1$ to n
n = no. of particles

for particles on the axis, $r = 0 \Rightarrow v = \omega r = 0 \Rightarrow$ stationary

Direction of ω :-



$\vec{v} = \vec{\omega} \times \vec{r}$; $v = \omega r_\perp$:-

$\omega \times r_\perp$ – acts along the tangent

Thus, $\vec{v} = \vec{\omega} \times \vec{r}$

Position vector of the particle w.r.t. origin $= \vec{OP} = \vec{r}$

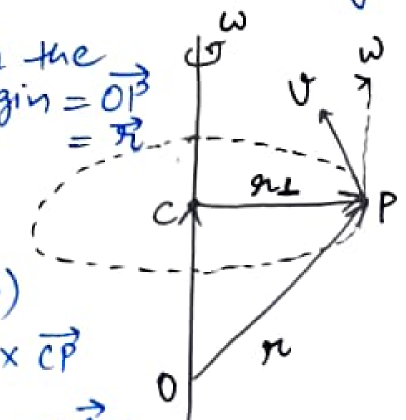
Now,

$\omega \times r = \omega \times \vec{OP}$

$= \omega \times (\vec{OC} + \vec{CP})$

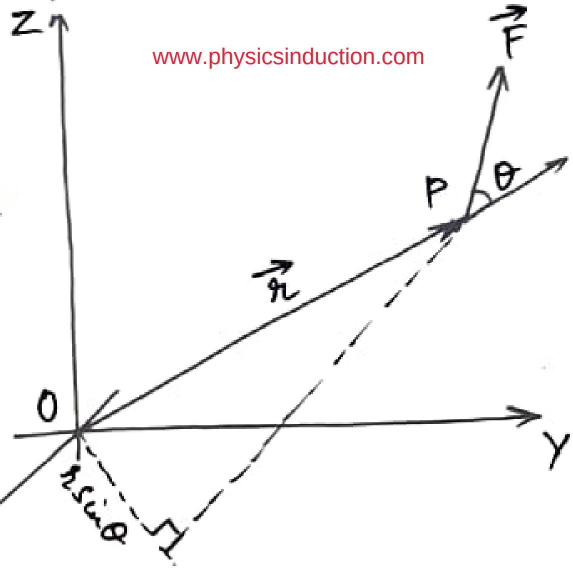
$= \underbrace{\omega \times \vec{OC}}_{\substack{\parallel \\ 0 \\ \text{(same direct}^n\text{)}}} + \omega \times \vec{CP}$

$= \omega \times \vec{r}_\perp$



ANGULAR ACCELERATION:- It's defined as the time rate of change of angular velocity. $\boxed{\alpha = \frac{d\omega}{dt}}$

TORQUE – MOMENT OF FORCE:- The rotational analogue of force is moment of force/torque/couple. It's the turning effect of the force about the fixed point/axis.



Torque, $\vec{\tau} = \vec{r} \times \vec{F}$
$= rF\sin\theta\,\hat{n}$

$\vec{r}$: ⊥r dist. of the line of action of force from the axis of rotation.

$\hat{n}$: Unit vector along τ

θ : Smaller angle b/w $\vec{r}$ & $\vec{F}$

direct^n of τ:- ⊥r to $\vec{r}$ & $\vec{F}$ (right hand screw rule)

Anticlockwise moments = +ve & vice-versa by convention.

Unit :- N-m

Dimensional formula :- $[ML^2T^{-2}]$

$\vec{\tau} = \vec{r} \times \vec{F} = rF\sin\theta = (r\sin\theta)F = r_\perp F$:-

$r_\perp = r\sin\theta$ is the perpendicular distance of the line of action of $\vec{F}$ from the origin.

$\vec{\tau} = \vec{r} \times \vec{F} = rF\sin\theta = r(F\sin\theta) = rF_\perp$:

$F_\perp = F\sin\theta$ is the component of $\vec{F}$ in the direction perpendicular to $\vec{r}$.

$\vec{\tau} = 0$ if $r = 0$ or $F = 0$ or $\theta = 0°$ or $180°$

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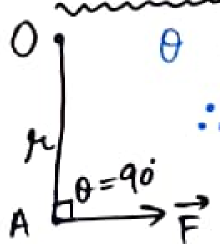
Cases :-

(i) Case-1 θ b/w $\vec{r}$ & $\vec{F} = 0°$

$\therefore \tau = rF\sin 0° = 0$

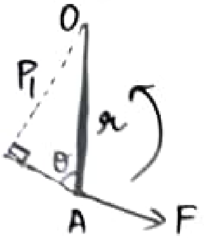
(ii) Case-2 :-



θ b/w $\vec{r}$ & $\vec{F} = 90°$

$\therefore \tau = rF\sin 90°$
$= rF$ (MAX^M)

(iii) **Case-3 :-**



$\tau = rF\sin\theta$

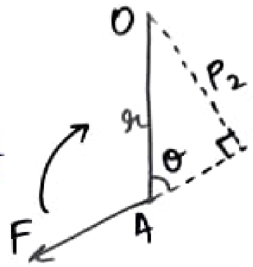
$= F(r\sin\theta)$

$= F \times P_1$

P_1 : ⊥r distance of line of action of force from O.

(anticlockwise)

(iv) **Case-4 :-**



$\tau = rF\sin\theta$

$= F(r\sin\theta)$

$= F \times P_2$

P_2 : ⊥r dist. of line of action of force from O.

(clockwise)

Expression for τ in Cartesian coordinates :- Physical meaning of τ

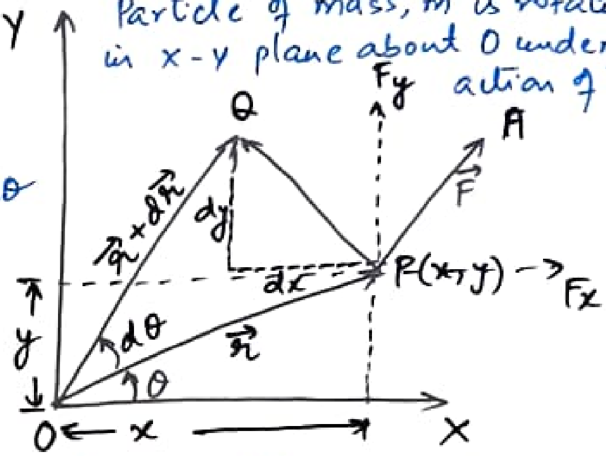
Particle of mass, m is rotating in x-y plane about O under the action of $\vec{F}$.

$\vec{F}$ is applied along $\vec{PA}$

Consider a particle at P at time, t & reaches Q at $(t+dt)$

$\vec{OP} = \vec{r}$, $\vec{OQ} = \vec{r} + d\vec{r}$, $\angle POQ = d\theta$

ΔPOQ : Δ law $\Rightarrow \vec{PQ} = d\vec{r}$

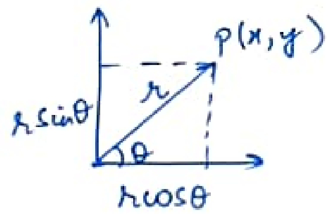


Small amount of W.D. in rotating the particle from P to Q is,

$$dW = \vec{F} \cdot d\vec{r}$$

$\vec{F} = F_x\hat{i} + F_y\hat{j}$, $d\vec{r} = dx\hat{i} + dy\hat{j}$

$\therefore \; dW = F_x dx + F_y dy \quad — (1)$



Now, $x = r\cos\theta$, $y = r\sin\theta$

differentiating w.r.t. θ we get,

$$dx = -r\sin\theta\, d\theta = -y\, d\theta \quad — (2)$$

$$\& \; dy = r\cos\theta\, d\theta = x\, d\theta \quad — (3)$$

Substituting (2) & (3) in (1),

$$dW = F_x(-y\,d\theta) + F_y(x\,d\theta)$$

$$= (xF_y - yF_x)\,d\theta = \tau\, d\theta \Rightarrow \boxed{\tau = xF_y - yF_x}$$

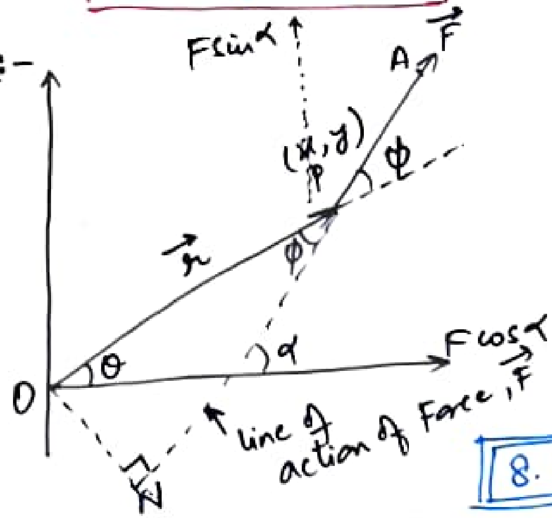
Expression for τ in polar coordinates :-

Consider a particle of mass, m rotating in X-Y plane about O, under $\vec{F}$ (along $\vec{PA}$)

line of action of force, F makes an angle, α with x-axis

$F_x = F\cos\alpha$, $F_y = F\sin\alpha$

$x = r\cos\theta$, $y = r\sin\theta$



As, $\tau = xF_y - yF_x$

$\therefore \tau = (r\cos\theta)(F\sin\alpha) - (r\sin\theta)(F\cos\alpha)$

$= rF(\cos\theta\sin\alpha - \sin\theta\cos\alpha)$

$= rF\sin(\alpha-\theta) = rF\sin\phi = F(r\sin\phi) = F(ON)$

Rectangular Components of τ:- If particle is rotating in x-y plane due to $\vec{F}$, τ will act on z-plane $\perp r$ to the x-y plane i.e., $\tau_x = 0 = \tau_y$.

$\vec{F}$ may rotate the body in 3-D $\Rightarrow$ 3-rectangular components

$$\vec{\tau} = \vec{r}\times\vec{F}$$

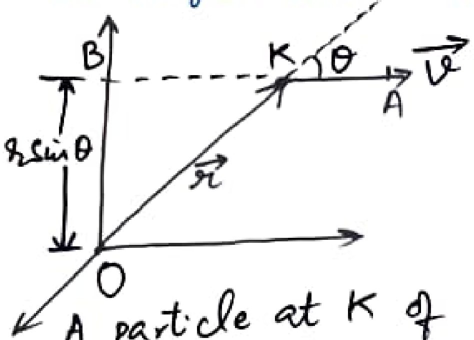
$$\Rightarrow \tau_x\hat{i} + \tau_y\hat{j} + \tau_z\hat{k} = (x\hat{i}+y\hat{j}+z\hat{k})\times(F_x\hat{i}+F_y\hat{j}+F_z\hat{k})$$

$$= (yF_z - zF_y)\hat{i} + (zF_x - xF_z)\hat{j} + (xF_y - yF_x)\hat{k}$$

$$\Rightarrow \boxed{\tau_x = yF_z - zF_y \;;\; \tau_y = zF_x - xF_z \;;\; \tau_z = xF_y - yF_x}$$

ANGULAR MOMENTUM:- Angular momentum is the rotational analogue of linear Momentum. It's a vector quantity.

Consider a particle of mass, m and linear momentum, $\vec{P}$ at a position, $\vec{r}$; relative to the origin, O. The angular momentum, $\vec{L}$ of the particle w.r.t. the origin, O, is defined to be, $\underline{\vec{L} = \vec{r}\times\vec{P}}$



A particle at K of mass, m; moving with velocity, $\vec{v}$ along $\vec{KA}$.

Angular Momentum, $\vec{L}$ 'bout O is;

$\vec{L} = \vec{OK}\times(m\vec{v})$

$= mv(OK\sin\theta)$

$= p(r\sin\theta)$

$= p(OB)$

linear Momentum

$\perp r$ distance of the line of motion from O.

$\vec{L} = \vec{r} \times \vec{P} = rP\sin\theta = r(P\sin\theta) = rP_\perp$:-

$P_\perp = P\sin\theta$ is the component of $\vec{P}$ in a direction perpendicular to $\vec{r}$.

$\vec{L} = \vec{r} \times \vec{P} = rP\sin\theta = (r\sin\theta)P = r_\perp P$:-

$r_\perp = r\sin\theta$ is the perpendicular distance of the directional line of $\vec{P}$ from the origin.

$L = 0$ if $P = 0$ or $r = 0$ or $\theta = 0°$ or $180°$

Conservation of Angular Momentum :- $\tau_{ext} = \frac{dL}{dt}$

Ist Method :- $\vec{L} = \vec{r} \times \vec{P}$

differentiating w.r.t. time

$$\frac{d\vec{L}}{dt} = \frac{d}{dt}(\vec{r} \times \vec{P})$$

$$= \frac{d\vec{r}}{dt} \times \vec{P} + \vec{r} \times \frac{d\vec{P}}{dt}$$ (Applying the product rule for differentiation)

$$= \vec{v} \times (m\vec{v}) + \vec{r} \times \vec{F}$$

$$= 0 + \vec{r} \times \vec{F} = \vec{\tau}$$

τ due to int f as well as ext f but int. τ add up to zero.

$$\therefore \vec{\tau}_{ext} = \frac{d\vec{L}}{dt}$$

IInd Method :- $L = I\omega$

$$\Rightarrow \frac{dL}{dt} = \frac{I\,d\omega}{dt} = I\alpha = \tau_{ext}$$

If total external torque, τ_{ext} on a system is zero, Its Angular momentum, $\vec{L}$ remains constant.

i.e., If $\tau_{ext} = 0$

$$\Rightarrow \frac{dL}{dt} = 0$$

$$\Rightarrow L = \text{constant}$$

Cartesian Coordinates of $\vec{\tau}$:-

As, $\tau_z = x F_y - y F_x$

$$= x\left[m \frac{dv_y}{dt}\right] - y\left[m\frac{dv_x}{dt}\right]$$

$$= m\left[x\frac{dv_y}{dt} - y\frac{dv_x}{dt}\right]$$

$$= m.\frac{d}{dt}\left(x v_y - y v_x\right)$$

$$= \frac{d}{dt}\left(x m v_y - y m v_x\right)$$

$$= \frac{d}{dt}\underbrace{\left(x P_y - y P_x\right)}_{L_z}$$

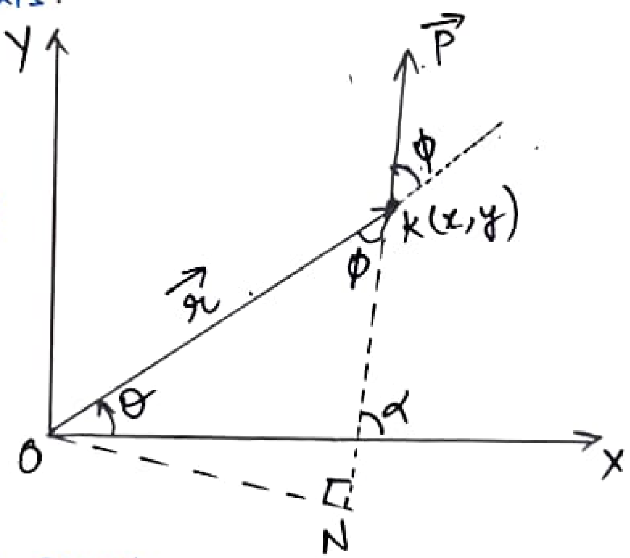
$\therefore$ $\boxed{L_z = x P_y - y P_x}$

|||ly, $\boxed{L_x = y P_z - z P_y \;,\quad L_y = z P_x - x P_z}$

Polar coordinates of $\vec{\tau}$: Physical Meaning

$K(x,y)$ - position of the particle of mass, m, linear momentum, $\vec{P}$; rotating in X-Y plane 'bout z-axis.

$\angle XOK = \theta$: $x = r\cos\theta$, $y = r\sin\theta$



Line of action of $\vec{P}$ makes an angle, α with OX, and angle, ϕ with $\vec{r}$.

$P_x = P\cos\alpha$, $\quad P_y = P\sin\alpha$

As,

$$L = x P_y - y P_x$$

$$= r\cos\theta . P\sin\alpha - r\sin\theta . P\cos\alpha$$

$$= rp\left[\sin\alpha\cos\theta - \cos\alpha\sin\theta\right]$$

$$= rp\sin(\alpha-\theta) = rp\sin\phi = p(r\sin\phi) = \underline{P(ON)}$$

As, $\underset{\uparrow}{L = P(ON)}$

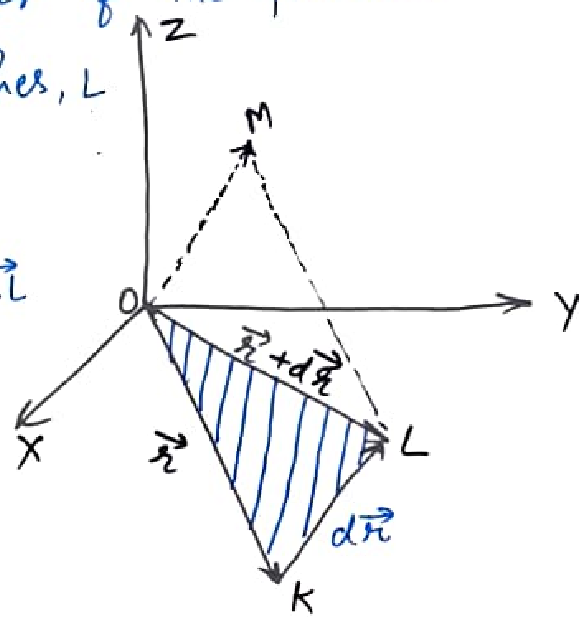
L is the product of P & ⊥r distance of line of action of $\vec{P}$ from the axis of rotation.

Geometrical Meaning of $\vec{L}$:- Consider a particle rotating in X-Y plane about an axis, OZ, at any time, t,
Let $\vec{OK} = \vec{r}$ be the position vector of the particle.

In small time, dt; it reaches, L

$\vec{OL} = \vec{r} + d\vec{r}$

Join KL, $\vec{KL} = d\vec{r}$



Draw $\vec{OM}$, equal & parallel to $\vec{KL}$

Area swept by the particle in small time, dt :

$$|d\vec{A}| = \text{Area of } \Delta OKL$$

$$= \frac{1}{2}[ar(\parallel gm\ OKLM)]$$

$$= \frac{1}{2}|\vec{OK} \times \vec{OM}| = \frac{1}{2}|\vec{r} \times d\vec{r}|$$

(divide by dt)

$$\Rightarrow \left|\frac{d\vec{A}}{dt}\right| = \frac{1}{2}\left|\vec{r} \times \frac{d\vec{r}}{dt}\right|$$

$$= \frac{1}{2}\left|\vec{r} \times \frac{\vec{P}}{m}\right| \qquad \left[\because \frac{d\vec{r}}{dt} = \vec{v} = \frac{\vec{P}}{m}\right]$$

$$= \frac{1}{2m}|\vec{r} \times \vec{P}|$$

$$= \frac{1}{2m}|\vec{L}| \qquad [\because \vec{L} = \vec{r} \times \vec{P}]$$

$$\Rightarrow \boxed{|\vec{L}| = 2m\left|\frac{d\vec{A}}{dt}\right|}$$

L of a particle about a given axis is twice the product of mass of the particle & areal velocity of posⁿ vector of the particle.

EQUILIBRIUM OF A RIGID BODY :- Translational Equilibrium + Rotational Equilibrium.

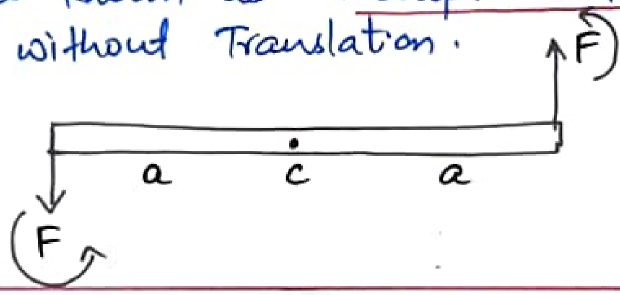
A rigid body is said to be in Mechanical Equilibrium. if both its linear & angular momentum remain constant, or equivalently, the body has neither linear acceleration nor angular acceleration.

$\therefore \sum_{i=1}^{n} F_i = 0$: condition for translational Equilibrium.

And, $\sum_{i=1}^{n} \tau_i = 0$: Condition for rotational Equilibrium.

COUPLE :-

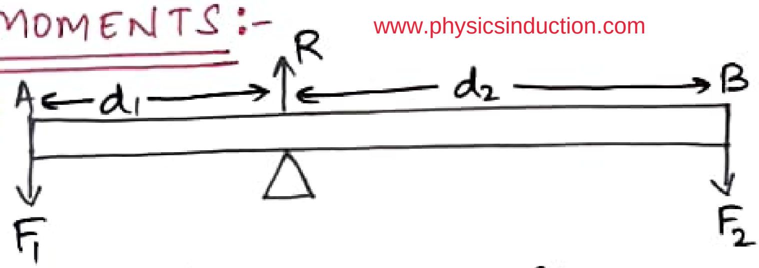
A pair of equal and opposite forces with different lines of action is known as a couple or Torque. A couple produces rotation without Translation.



PRINCIPLE OF MOMENTS :-

Anticlockwise Moments – +ve

Clockwise Moments – -ve



R acts at the Fulcrum & it has zero moment about the Fulcrum

Lever Force, F_1 is usually some weight to be lifted. It's called the load. & its distance from the fulcrum, d_1 is called the Load Arm. Force, F_2 is the effort applied to lift the load. the distance, d_2 of the effort from the fulcrum is the effort Arm.

* For Translational Equilibrium : $R - F_1 - F_2 = 0$

* For Rotational Equilibrium :- The sum of moments must be zero.

i.e., $d_1 F_1 - d_2 F_2 = 0$

$\Rightarrow \quad d_1 F_1 = d_2 F_2$

$\Rightarrow$ Load Arm x load = Effort Arm x Effort

MECHANICAL ADVANTAGE :- $M.A. = \frac{F_1}{F_2}$

As, $d_1 F_1 = d_2 F_2$

$$\Rightarrow \frac{F_1}{F_2} = \frac{d_2}{d_1}$$

If $d_2 > d_1$ (Effort Arm > Load Arm) $\Rightarrow$ M.A. > 1

$\Rightarrow$ Small effort can be used to lift a large load.

CENTRE OF GRAVITY :- The centre of gravity of an extended body is that point, where the total gravitational torque on the body is zero.

Torque on the ith particle about the CG, due to the force of gravity, $\tau_g = \sum_i \tau_i = \sum r_i \times m_i g = 0$

$$\because g \neq 0 \Rightarrow \sum_i m_i r_i = 0 \quad \text{— (1)}$$

(* pos^n vectors, r_i are taken w.r.t. the CG)

As, pos^n vector of C.M., $\vec{r} = \frac{\sum m_i \vec{r_i}}{M}$

from (1), if $\sum m_i r_i = 0 \Rightarrow \vec{r} = 0 \Rightarrow$ origin must be the CM.

$\Rightarrow$ CG coincides with CM in uniform gravity or gravity-free space.

MOMENT OF INERTIA :- M.I. is analogous of mass in linear Motion.
(I)

M.I. - Rotational Inertia :- Body opposes any change in its state of Uniform Rotation.

Def^n: It's the sum of the products of masses & square of respective ⊥r distance from the axis of rotation.

$$\boxed{I = \sum_{i=1}^{n} m_i r_i^2}$$

I depends on :- (i) Position of axis of Rotation (iii) Shape (Body)
(ii) Orientation of axis of Rotation (iv) Size (Body)
(V) Distribution of Mass 'bout the axis of rotation.

Unit of M.I, I :- $kg\,m^2$, $g\,cm^2$ Dimensions :- $[M^1 L^2 T^0]$

$\tau = I\alpha$: $\therefore$ M.I. 'bout a given axis, is torque on the body, rotating with unit angular acceleration 'bout it.

i.e, $\boxed{\tau = I \text{ at } \alpha = 1}$

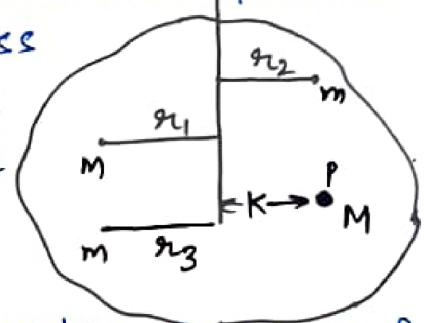
K.E. $= \frac{1}{2} I\omega^2$:- If $\omega = 1$ then, K.E. $= \frac{I}{2}$ $\Rightarrow \boxed{I = 2\times K.E. \text{ at } \omega = 1}$

As, $I = \sum_i m_i r_i^2 = MK^2$

where, $M = m_1 + m_2 + ----$

& K = effective distance of the particle from the axis = Radius of gyration.

RADIUS OF GYRATION :- Perpendicular distance from the axis of rotation, at which if its entire mass be supposed to be concentrated, Its M.I. about the given axis, would be the same, as with the actual distribution of mass.



Consider a Rigid body, consisting of n, no. of particles, each of mass, m. Let $r_1, r_2, ---, r_n$ be their perpendicular distance from the axis of rotation.

$$I = mr_1^2 + mr_2^2 + mr_3^2 + ------ + mr_n^2 \quad \text{(by def}^n\text{)}$$

$$= m(r_1^2 + r_2^2 + r_3^2 + ------- + r_n^2)$$

$$= m\times n \left[\frac{r_1^2 + r_2^2 + r_3^2 + ---- + r_n^2}{n}\right]$$

$$= M\left[\frac{r_1^2 + r_2^2 + r_3^2 + ----- + r_n^2}{n}\right] \quad \text{—(1)} \quad \left[\text{where } M = m\times n = \text{total mass}\right]$$

If total mass were concentrated at P at a $\perp r$ distance, K from the axis, $I' = MK^2$ —— (2)

from (1) & (2) : $I' = I$

$$\therefore \boxed{K = \sqrt{\frac{r_1^2 + r_2^2 + ---- + r_n^2}{n}}}$$

* When a body is rotating in X-Y Plane:

$\vec{r} = x\hat{i} + y\hat{j} \Rightarrow r^2 = x^2 + y^2$ & $I = \Sigma m(x^2+y^2)$

* When a body doesn't consist of discrete particles & has a continuous distribution of mass:

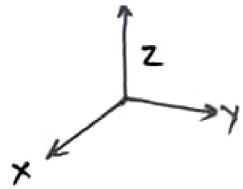
$I = \int r^2\, dm$, dm - small element at a ⊥r distance, r from the axis.

THEOREMS ON M.I. :-

(When M.I. of a body about any given axis is known, we can calculate M.I. about another axis)

Theorem of perpendicular axes:-

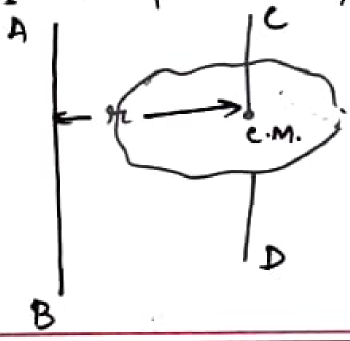
M.I. of a plane lamina 'bout an axis ⊥r to the plane of lamina is equal to the sum of M.I. of the lamina 'bout an axis ⊥r to each other.



$$\underline{I_Z = I_X + I_Y}$$

Theorem of parallel axes:-

M.I. of a body about an axis is equal to the M.I. about a parallel axis through its CM plus the product of mass & square of the distance b/w two axes.



$$\underline{I_{AB} = I_{CD} + Mr^2}$$

KINEMATICS OF ROTATIONAL MOTION ABOUT A FIXED AXIS:

$\omega = \frac{d\theta}{dt}$, $\alpha = \frac{d\omega}{dt}$

(i) $\omega = \omega_0 + \alpha t$, (ii) $\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2$, (iii) $\omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0)$

DYNAMICS OF ROTATIONAL MOTION ABOUT A FIXED AXIS:-

$dW = \tau\, d\theta$

Instantaneous Power, $P = \frac{dW}{dt} = \frac{d(\tau d\theta)}{dt} = \tau \frac{d\theta}{dt} = \tau\omega$

$$\boxed{P = \tau\omega}$$

K.E. $= \boxed{\frac{1}{2} I\omega^2}$

The rate of increase of K.E :

$$\frac{d}{dt}\left(\frac{1}{2}I\omega^2\right) = \frac{1}{2}I(2\omega)\frac{d\omega}{dt} = I\omega\frac{d\omega}{dt} = \underline{I\omega\alpha}$$

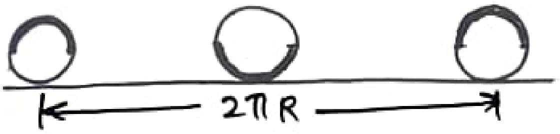
As, dW = change in K.E.

$\therefore$ Rates of W.D. and of increase in K.E. are equal

i.e., $\frac{dW}{dt} = \frac{d}{dt}\left(\frac{1}{2}I\omega^2\right)$

$$\Rightarrow \tau\omega = I\omega\alpha \Rightarrow \boxed{\tau = I\alpha}$$

ROLLING MOTION :-



During one rotation, it has rotated thro' an angle, 2π centre of the wheel goes in a straight line and during one rotation, it moves a distance of $2\pi R$ & spoke makes an angle of 2π.

During a short time interval, Δt, wheel moves a distance, Δx & spoke rotates by $\Delta\theta$. Thus, wheel rotates and at the same time, moves forward. $\Delta x = R\Delta\theta$

$$\Rightarrow \frac{\Delta x}{\Delta t} = R\frac{\Delta\theta}{\Delta t} \Rightarrow \boxed{v = R\omega} \rightarrow \text{Case of Pure Rolling}$$

In Pure Rolling,

Velocity of the contact point $= \underline{0}$

" " " C.M., $v_{cm} = \underline{R\omega}$

" " " topmost point, $v_{top} = 2R\omega = \underline{2v_{cm}}$

K.E. OF A BODY IN A COMBINED ROTATION & TRANSLATION :-

There are two cases :-

(i) C.M. of the rotating body has zero linear velocity.

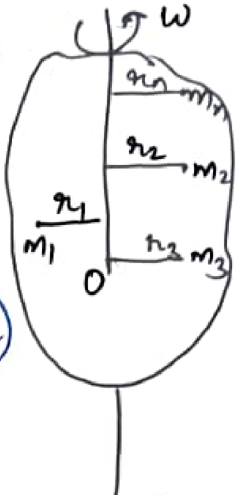
(ii) C.M. of the rotating body has also a linear velocity.

(i) C.M. of the rotating body has zero linear Velocity :-

M : mass of the rigid body, moving with angular velocity, ω

& O : C.M.

Let body is consisting of n, number of particles of of masses $m_1, m_2, ----, m_n$ and distance from the axis of rotation as $r_1, r_2, ----, r_n$



$$\text{Total K.E.} = \frac{1}{2}\left(m_1 v_1^2 + m_2 v_2^2 + ---- + m_n v_n^2\right)$$

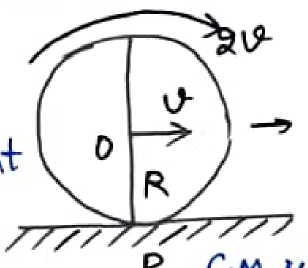
$$\therefore \text{K.E.} = \frac{1}{2}\omega^2\left(m_1 r_1^2 + m_2 r_2^2 + ---- + m_n r_n^2\right) \quad \because v = \omega r)$$

$$= \frac{1}{2}\omega^2 \sum_{i=1}^{n} m_i r_i^2$$

$$= \frac{1}{2}\omega^2 M K^2$$: M - Total mass, K - Radius of gyration.

(ii) C.M. of the rotating body has also linear Velocity :-

(a) Case-1 :- Case of a body rolling along a plane surface



At any instant, body is at P, particles of the body have the same angular velocity w.r.t. P. The motion of the body is thus equivalent to one of the pure rotation about the axis through P.

K.E. (translational Motion) = K.E. (Rotational Motion)

$$\therefore \quad \text{K.E.} = \frac{1}{2} I_P \omega^2$$

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$$\text{where,} \quad I_P = I_{cm} + M(R)^2$$

$$\therefore \quad \text{K.E.} = \frac{1}{2}\left(I_{cm} + MR^2\right)\omega^2$$

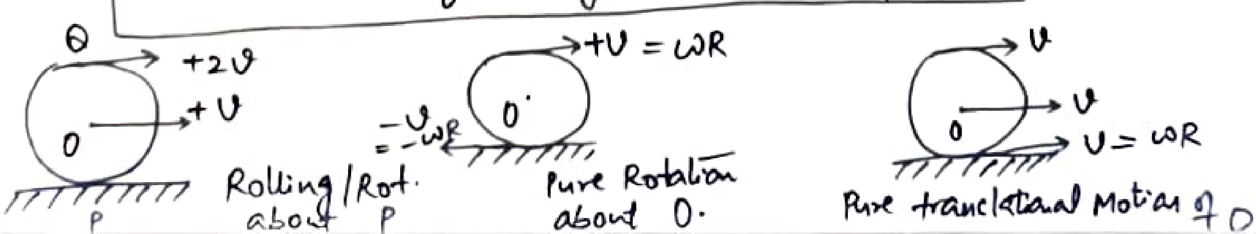
$$= \frac{1}{2} I_{cm}\omega^2 + \frac{1}{2} MR^2\omega^2 \qquad [\because v = \omega R]$$

$$= \frac{1}{2} I_{cm}\omega^2 + \frac{1}{2} M v^2$$

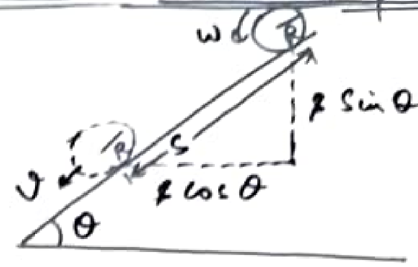
$$= \frac{1}{2}\left[MK^2\right]\left[\frac{v}{R}\right]^2 + \frac{1}{2} M v^2 \qquad \left[\because I_{cm} = MK^2, \omega = \frac{v}{R}\right]$$

$$= \frac{1}{2} M v^2 \left[\frac{K^2}{R^2} + 1\right]$$

$$\therefore \boxed{\text{K.E. of Rolling Body} = \frac{1}{2} M v^2 \left[\frac{K^2}{R^2} + 1\right]}$$



(b) **Case-2 : Case of a body rolling down an inclined plane - its acceleration along the plane** –



v: Velocity acquired by the body in covering the distance, s.

P.E. lost by the body $= Mg\,s\sin\theta$

P.E. = Total K.E. = K.E. of Rotation + K.E. of translation.

$$\therefore\ Mg\,s\sin\theta = \frac{1}{2}I\omega^2 + \frac{1}{2}Mv^2$$

$$= \frac{1}{2}[MK^2]\left[\frac{v}{R}\right]^2 + \frac{1}{2}Mv^2$$

$$= \frac{1}{2}Mv^2\left[\frac{K^2}{R^2}+1\right]$$

$$\Rightarrow\ v^2\left[\frac{K^2}{R^2}+1\right] = 2g\sin\theta\cdot s$$

$$\Rightarrow\ v^2\left[\frac{K^2+R^2}{R^2}\right] = 2g\sin\theta\cdot s$$

$$\Rightarrow\ v^2 = \frac{2R^2}{K^2+R^2}\,g\sin\theta\cdot s$$

As the body starts from rest, $u=0 \Rightarrow v^2 = 2as \Rightarrow a = \frac{v^2}{2s}$

$$\therefore\ \boxed{a = \frac{R^2}{K^2+R^2}\,g\sin\theta}$$